PROBLEM:
The time-averaged potential of a neutral hydrogen atom is given by
\[
\Phi = -\frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} \left( 1 + \frac{\alpha r}{2} \right)
\]
where \( q \) is the magnitude of the electronic charge, and \( \alpha^{-1} = a_0/2 \), \( a_0 \) being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

SOLUTION:
The Poisson equation links charge densities and the electric scalar potential that they create. We use it here to find the charge density. We must perform a straightforward differentiation in spherical coordinates.
\[
\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}
\]
Expand this in spherical coordinates:
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = -\frac{\rho}{\epsilon_0}
\]
The potential is spherically symmetric, so that the potential depends only on the radial coordinate - the partial derivatives of the potential are all zero, except for the one with respect to the radial component.
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = -\frac{\rho}{\epsilon_0}
\]
Evaluate the equation explicitly:
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \left[ \frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} \left( 1 + \frac{\alpha r}{2} \right) \right] \right) = -\frac{\rho}{\epsilon_0}
\]
\[
\frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \left[ \frac{e^{-\alpha r}}{r} \right] \right) + r^2 \frac{\partial}{\partial r} \left[ \frac{\alpha}{2} e^{-\alpha r} \right] = -\frac{\rho}{\epsilon_0}
\]
\[
\frac{q}{4\pi \epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \left[ \frac{1}{r} \frac{\partial}{\partial r} e^{-ar} + e^{-ar} \frac{1}{\partial r} \frac{\partial}{\partial r} \right] - \frac{\alpha^2}{2} r^2 e^{-ar} \right) = -\frac{\rho}{\epsilon_0}
\]

\[
\frac{q}{4\pi \epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left( -\alpha r e^{-ar} + r^2 e^{-ar} \frac{1}{\partial r} \frac{\partial}{\partial r} - \frac{\alpha^2}{2} r^2 e^{-ar} \right) = -\frac{\rho}{\epsilon_0}
\]

\[
\frac{q}{4\pi \epsilon_0} \left( -\alpha e^{-ar} + \frac{\alpha^3}{2} e^{-ar} \right) + \frac{q}{4\pi \epsilon_0} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 e^{-ar} \frac{1}{\partial r} \frac{\partial}{\partial r} \right) = -\frac{\rho}{\epsilon_0}
\]

\[
\rho = -\frac{q \alpha^3}{8\pi} e^{-ar} - \frac{e^{-ar}}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{1}{\partial r} \frac{\partial}{\partial r} \right) \quad \text{if } r \approx 0
\]

\[
\rho = -\frac{q \alpha^3}{8\pi} e^{-ar} - \frac{e^{-ar}}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{1}{\partial r} \frac{\partial}{\partial r} \right) \quad \text{if } r > 0
\]

Away from the origin, \( \frac{1}{r} \) does not blow up and the derivatives can be evaluated normally. The last term ends up equating to zero, so that our equations now becomes:

\[
\rho = -\frac{q \alpha^3}{8\pi} e^{-ar} - \frac{e^{-ar}}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{1}{\partial r} \frac{\partial}{\partial r} \right) \quad \text{if } r \approx 0
\]

\[
\rho = -\frac{q \alpha^3}{8\pi} e^{-ar} \quad \text{if } r > 0
\]

At \( r \approx 0 \), we have \( e^{-ar} = 1 \) so that the two cases become:

\[
\rho = -\frac{q \alpha^3}{8\pi} - \frac{q}{4\pi} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{1}{\partial r} \frac{\partial}{\partial r} \right) \quad \text{if } r \approx 0
\]

\[
\rho = -\frac{q \alpha^3}{8\pi} e^{-ar} \quad \text{if } r > 0
\]

Now use the relation:

\[
\nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta(r)
\]

which when evaluated explicitly becomes:
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \right) \right) = -4\pi \delta(r)
\]

Plug this into the above set of equations:

\[
\rho = -\frac{q \alpha^3}{8\pi} + q \delta(r) \quad \text{if} \quad r \approx 0
\]

\[
\rho = -\frac{q \alpha^3}{8\pi} e^{-\alpha r} \quad \text{if} \quad r > 0
\]

Because the delta function is zero everywhere except at the origin, and because the first term of the first equation is just the specific \( r = 0 \) form of the first term of the second equation, the two cases can be combined into one case:

\[
\rho = -\frac{q \alpha^3}{8\pi} e^{-\alpha r} + q \delta(r) \quad \text{for all} \quad r
\]

This corresponds physically to a positive point charge at the origin with one unit of elementary charge, and a finite cloud of negative charge that decays exponentially, but contains a total charge of one unit of elementary charge.

From a time-averaged perspective then, hydrogen in the ground state contains a positive point charge at the center and a circular cloud of negative charge. This is of course only useful for conceptualization purposes, because at atomic sizes the system behaves quantum mechanically, not classically.