Lecture 6 Notes, Electromagnetic Theory I

Dr. Christopher S. Baird
University of Massachusetts Lowell

1. Associated Legendre Polynomials

- We now return to solving the Laplace equation in spherical coordinates when there is no azimuthal symmetry by solving the full Legendre equation for $m = 0$ and $m \neq 0$:

$$\frac{d}{dx}\left[(1-x^2)\frac{d P_l^m(x)}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P_l^m(x)=0 \quad \text{where } x=\cos\theta$$

- Once this equation is solved, the general solution for the Laplace equation in spherical coordinates will have the form:

$$\begin{aligned}\Phi(r,\theta,\phi)=&\sum_l (A_l r^l+B_l r^{-l-1})(A_{m0}+B_{m0}\phi)P_l^{m=0}(\cos\theta)\\ &+\sum_{m\neq 0,l}(A_l r^l+B_l r^{-l-1})(A_m e^{im\phi}+B_m e^{-im\phi})P_l^m(\cos\theta)\end{aligned}$$

- Encouraged by the Rodrigues' form of the solution to the $m = 0$ Legendre equation, we try a solution of the form $P_l^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}g(x)$ and substitute in:

$$\begin{aligned}0=&-m\,x(1-x^2)^{m/2}\left[\frac{d^{m+1}}{dx^{m+1}}g(x)\right]+m^2\frac{d^m}{dx^m}g(x)x^2(1-x^2)^{m/2-1}-m\frac{d^m}{dx^m}g(x)(1-x^2)^{m/2}\\ &+(1-x^2)^{m/2+1}\left[\frac{d^{m+2}}{dx^{m+2}}g(x)\right]-(m+2)x(1-x^2)^{m/2}\frac{d^{m+1}}{dx^{m+1}}g(x)\\ &+[l(l+1)](1-x^2)^{m/2}\frac{d^m}{dx^m}g(x)-m^2(1-x^2)^{m/2-1}\frac{d^m}{dx^m}g(x)\end{aligned}$$

- Collect all similar terms:

$$0=(1-x^2)\left[\frac{d^m}{dx^m}\frac{d^2}{dx^2}g(x)\right]-2[m+1]x\frac{d^{m+1}}{dx^{m+1}}g(x)+[l(l+1)-m(m+1)]\frac{d^m}{dx^m}g(x)$$

- We want to move as much as possible inside the derivatives. We can do this by using the product rule.

- The product rule used m times states: $\boxed{\frac{d^m}{dx^m}(uv)=\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\left[\frac{d^{m-k}}{dx^{m-k}}u\right]\left[\frac{d^k}{dx^k}v\right]}$

- Let us try to apply this to rearrange the first term in the differential equation.

- If $u=1-x^2$ then $\frac{d}{dx}u=-2x$, $\frac{d^2}{dx^2}=-2$ and all higher derivatives in the expansion are zero so that:

$$\frac{d^m}{dx^m}((1-x^2)v)=(1-x^2)\left[\frac{d^m}{dx^m}v\right]-2mx\left[\frac{d^{m-1}}{dx^{m-1}}v\right]-m(m-1)\left[\frac{d^{m-2}}{dx^{m-2}}v\right]$$

- Moving terms around:

$$(1-x^2)\left[\frac{d^m}{dx^m}v\right]=\frac{d^m}{dx^m}((1-x^2)v)+2mx\left[\frac{d^{m-1}}{dx^{m-1}}v\right]+m(m-1)\left[\frac{d^{m-2}}{dx^{m-2}}v\right]$$

- Now if we set $v=\frac{d^2}{dx^2}g(x)$ this product rule expansion becomes:

$$(1-x^2)\left[\frac{d^m}{dx^m}\frac{d^2}{dx^2}g(x)\right]=\frac{d^m}{dx^m}((1-x^2)\frac{d^2}{dx^2}g(x))+2mx\left[\frac{d^{m+1}}{dx^{m+1}}g(x)\right]+m(m-1)\left[\frac{d^m}{dx^m}g(x)\right]$$

- We can use this product rule to rearrange the first term of the Legendre equation so it becomes:

$$0=\frac{d^m}{dx^m}\left[(1-x^2)\frac{d^2}{dx^2}g(x)\right]-2x\frac{d^{m+1}}{dx^{m+1}}g(x)+l(l+1)\frac{d^m}{dx^m}g(x)-2m\left[\frac{d^m}{dx^m}g(x)\right]$$

- Now we want to use the product rule expansion on the second term of the Legendre equation. With $u=x$, $\frac{du}{dx}=1$, all higher derivatives are zero and $v=\frac{d}{dx}g(x)$, the general product rule becomes:

$$\frac{d^m}{dx^m}\left[x\frac{d}{dx}g(x)\right]=x\left[\frac{d^{m+1}}{dx^{m+1}}g(x)\right]+m\left[\frac{d^m}{dx^m}g(x)\right]$$

- After rearranging:

$$x\left[\frac{d^{m+1}}{dx^{m+1}}g(x)\right]=\frac{d^m}{dx^m}\left[x\frac{d}{dx}g(x)\right]-m\left[\frac{d^m}{dx^m}g(x)\right]$$

- Substitute in:

$$0=\frac{d^m}{dx^m}\left[(1-x^2)\frac{d^2}{dx^2}g(x)-2\left[x\frac{d}{dx}g(x)\right]+l(l+1)g(x)\right]$$

$$\boxed{0=\frac{d^m}{dx^m}\left[\frac{d}{dx}\left((1-x^2)\frac{d}{dx}g(x)\right)+l(l+1)g(x)\right]}$$

- It is worth noting that if this is true for *m* = 0, then it will automatically be true for all *m*. So we can set $m = 0$:

$$0 = \frac{d}{dx}\left((1-x^2)\frac{d}{dx}g(x)\right) + l(l+1)g(x)$$

- Now this is just the $m = 0$ Legendre equation, which we have already solved. We found the solutions to be the ordinary Legendre polynomials, $P_l(x)$, so that $g(x) = P_l(x)$.

- We now have the solution to the full Legendre equation:

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

- There is an arbitrary phase factor that we have assumed to be one for simplicity, but is conventionally set to $(-1)^m$, so that the associated Legendre functions become:

$$\boxed{P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)}$$

- Or written explicitly using Rodrigues' equation:

$$\boxed{P_l^m(x) = \frac{(-1)^m}{2^l\, l\,!}(1-x^2)^{m/2}\frac{d^{m+l}}{dx^{m+l}}(x^2-1)^l}$$

- It is worth noting upon examination of the equation

$$0 = \frac{d^m}{dx^m}\left[\frac{d}{dx}\left((1-x^2)\frac{d}{dx}g(x)\right) + l(l+1)g(x)\right]$$

that the highest power of x inside the brackets is l, so that m cannot be greater than l, otherwise the equation would be trivially satisfied and thus there would be no unique solution. In summary, both l and m are integers and the possible m are $-l$, $-(l-1)$, ..., 0, ..., $(l-1)$, l.
- Several useful mathematical relations involving the associated Legendre functions can be found (the derivations are left to the interested student).

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

$$(l-m+1)P_{l+1}^m(x) = (2l+1)x\,P_l^m(x) - (l+m)P_{l-1}^m(x)$$

$$\sqrt{1-x^2}\,P_l^{m+1}(x) = (l-m)x\,P_l^m(x) - (l+m)P_{l-1}^m(x)$$

$$(x^2-1)\frac{d}{dx}P_l^m(x) = l\,x\,P_l^m(x) - (l+m)P_{l-1}^m(x)$$

- For fixed m, the Legendre functions form an orthogonal set:

$$\int_{-1}^{1} P_{l'}^{m}(x) P_{l}^{m}(x)\,dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l'l}$$

2. Spherical Harmonics

- With the full Legendre equation now solved, the general solution of the Laplace equation in spherical coordinates has been found:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})(A_{m0} + B_{m0}\phi) P_l^{m=0}(\cos\theta)$$
$$+ \sum_{l=0}^{\infty}\sum_{m=1}^{l}(A_l r^l + B_l r^{-l-1})(A_m e^{im\phi} + B_m e^{-im\phi}) P_l^m(\cos\theta)$$

where P_l^m are the associated Legendre functions.

- Typically, most problems require a valid solution on the full range of ϕ . The single-valued requirement in this case forces $B_{m0} = 0$, and the $m = 0$ term can now be included in the sum with the rest of the terms:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}(A_l r^l + B_l r^{-l-1})(A_m e^{im\phi} + B_m e^{-im\phi}) P_l^m(\cos\theta)$$

- An alternate way of presenting this is to let m sum from $-l$ to l and thus combine the A_m and B_m terms with the other constants:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(A_{l,m} r^l + B_{l,m} r^{-l-1}) e^{im\phi} P_l^m(\cos\theta)$$

- The normalizing terms are included in the undetermined constants A_{lm} and B_{lm}. When a boundary condition is applied, these constants become specified and the normalizing terms result naturally. However, to make the math cleaner, we can explicitly pull the normalizing terms out of the constants in advance.

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(A_{l,m} r^l + B_{l,m} r^{-l-1})\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\phi} P_l^m(\cos\theta)$$

- It is now evident that $P^l{}_m(\cos\theta)$ times its normalization term forms a complete set of orthonormal functions in the variable θ, indexed by l for a fixed m, and $e^{im\varphi}$ times its normalization term forms a complete set of orthonormal functions in the variable φ, indexed by m. The product of both thus creates orthonormal functions that spans all angles of the unit circle. These are known as "spherical harmonics" Y_{lm}:

$$\boxed{Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}} e^{im\phi} P_l^m(\cos\theta)}$$ so that

$$\boxed{\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{l,m}r^{l}+B_{l,m}r^{-l-1}\right)Y_{lm}(\theta,\phi)}$$

- The orthonormality of the spherical harmonics explicitly means that:

$$\int_0^{2\pi}\int_{-1}^{1}Y^*_{l'm'}(x,\phi)Y_{lm}(x,\phi)\,dx\,d\phi=\delta_{l'l}\delta_{m'm}\quad\text{where } x=\cos\theta$$

or $$\boxed{\int_0^{2\pi}\int_0^{\pi}Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi)\sin\theta\,d\theta\,d\phi=\delta_{l'l}\delta_{m'm}}$$

- The validity of this equation can be checked trivially by expressing the spherical harmonics in terms of its definition and applying the orthonormality of the associated Legendre functions and the complex exponentials.
- Using the definitions of the spherical harmonics, the patient student can work out the explicit analytic form for any given *l* and *m*. The lowest-order spherical harmonics are especially simple and are typically tabulated in textbooks.
- Because the spherical harmonics form an orthonormal set of equations, an arbitrary function $f(\theta,\phi)$ can be expanded in terms of spherical harmonics:

$$\boxed{f(\theta,\phi)=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_{lm}(\theta,\phi)}\quad\text{where}\quad\boxed{A_{lm}=\int_0^{2\pi}\int_0^{\pi}f(\theta,\phi)Y^*_{lm}(\theta,\phi)\sin\theta\,d\theta\,d\phi}$$

- There are several useful special cases for spherical harmonics that we should keep in mind.
- If *m* = 0, the spherical harmonic does not depend on the azimuthal angle and the associated Legendre function reduces down to a Legendre polynomial:

$$\boxed{Y_{l,0}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)}$$

- If *m* = *l*, the spherical harmonic's dependence on polar angle becomes especially simple, just the sine function to the *m*th power, so there is only one maximum in this direction:

$$\boxed{Y_{m,m}(\theta,\phi)=\frac{(-1)^m}{2^m m!}\sqrt{\frac{(2m+1)(2m)!}{4\pi}}e^{im\phi}\sin^m\theta}$$

- The spherical harmonics with negative powers of *m* are trivially related to those with positive powers:

$$\boxed{Y_{l,-m}(\theta,\phi)=(-1)^m Y^*_{l,m}(\theta,\phi)}$$

- Regions of interest include the positive and negative z axis, where the spherical harmonics simplify:

$$\boxed{Y_{lm}(0,\phi)=\left\{\begin{matrix}0 \text{ if } m\neq 0\\ \sqrt{\frac{2l+1}{4\pi}} \text{ if } m=0\end{matrix}\right\}} \quad \text{and} \quad \boxed{Y_{lm}(\pi,\phi)=\left\{\begin{matrix}0 \text{ if } m\neq 0\\ \sqrt{\frac{2l+1}{4\pi}}(-1)^l \text{ if } m=0\end{matrix}\right\}}$$

3. Boundary Value Problems in Spherical Coordinates

- Consider the problem where there is a region of space without any charges, bounded by a sphere with radius r_0 centered at the origin on which there is a potential $\Phi(r=r_{0,}\theta,\phi)=V(\theta,\phi)$. We want to know the potential inside the sphere. We use the general solution to the Laplace equation in spherical coordinates:

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(A_{lm}r^l+B_{lm}r^{-l-1})Y_{lm}(\theta,\phi)$$

- The region of interest includes the origin, thus $B_{lm}=0$ to keep the solution from blowing up there:

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}r^l Y_{lm}(\theta,\phi)$$

- Now apply the boundary condition:

$$V(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}r_0^l Y_{lm}(\theta,\phi)$$

- This is an expansion in spherical harmonics, and the constants are thus:

$$A_{lm}r_0^l=\int_0^{2\pi}\int_0^{\pi}V(\theta,\phi)Y_{lm}^*(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

- The solution is then:

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}r^l Y_{lm}(\theta,\phi) \quad \text{where} \quad A_{lm}=r_0^{-l}\int_0^{2\pi}\int_0^{\pi}V(\theta,\phi)Y_{lm}^*(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

- Or written in a more intuitive way by redefining the constant:

$$\boxed{\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}\left(\frac{r}{r_0}\right)^l Y_{lm}(\theta,\phi)} \quad \text{where} \quad \boxed{A_{lm}=\int_0^{2\pi}\int_0^{\pi}V(\theta,\phi)Y_{lm}^*(\theta,\phi)\sin\theta\, d\theta\, d\phi}$$

4. The Addition Theorem for Spherical Harmonics

- Consider two coordinate vectors (*r*, *θ*, *φ*) and (*r*', *θ*', *φ*') which have the angle *γ* between them.
- The ordinary Legendre polynomial as a function of the angle between them, $P_l(\cos\gamma)$ can be expanded in spherical harmonics of the coordinate vectors:

$$P_l(\cos\gamma)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

- This is known as the addition theorem.
- Previously, we derived the potential at **r** in spherical coordinates that results from a unit charge at some point $\mathbf{r}_0$ *on the z-axis* as:

$$\Phi(r,\theta,\phi)=\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\sum_{l=0}^{\infty}\frac{r^l}{r_0^{l+1}}P_l(\cos\theta)\quad \text{if } r<r_0 \text{ and }\quad \frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\sum_{l=0}^{\infty}\frac{r_0^l}{r^{l+1}}P_l(\cos\theta)\quad \text{if } r>r_0$$

- The equations still hold when the point $\mathbf{r}_0$ is off the *z*-axis if the Legendre polynomials become functions of the angle *γ* between the observation point **r** and the source point $\mathbf{r}_0$.

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\sum_{l=0}^{\infty}\frac{r^l}{r_0^{l+1}}P_l(\cos\gamma)\quad \text{if } r<r_0 \text{ and }\quad \frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\sum_{l=0}^{\infty}\frac{r_0^l}{r^{l+1}}P_l(\cos\gamma)\quad \text{if } r>r_0$$

- This equation is not terribly useful because the angle *γ* depends on the spherical coordinates.
- We can use the addition theorem to expand the ordinary Legendre polynomial so that the expression is a function of spherical coordinates rather than a function of the angle between coordinate vectors:

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{r_0^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}\quad \text{if } r<r_0 \text{ and}$$

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_0^l}{r^{l+1}}\frac{1}{2l+1}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}\quad \text{if } r>r_0$$

- This will be useful if we want to know the potential produced by multiple point charges in explicit coordinates.

5. The Laplace Equation in Cylindrical Coordinates

- Earlier, we solved the Laplace equation in cylindrical coordinates for the special case of when the boundary conditions are uniform in the *z*-dimension, and the problem reduced to polar coordinates.
- We now revisit the problem for when the boundary conditions are not uniform in the *z*-dimension.
- The Laplace equation in cylindrical coordinates is:

$$\nabla^2\Phi=0 \quad\rightarrow\quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

- Use the method of separation of variables by trying a solution of the form:

$$\Phi(r,\phi,z)=R(\rho)Q(\phi)Z(z)$$

and substituting it in:

$$Q(\phi)Z(z)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)+R(\rho)Z(z)\frac{1}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial\phi^2}+R(\rho)Q(\phi)\frac{\partial^2 Z(z)}{\partial z^2}=0$$

- Divide by $R(\rho)Q(\phi)Z(z)$:

$$\frac{1}{R(\rho)}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)+\frac{1}{Q(\phi)}\frac{1}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial\phi^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}=0$$

- The last term depends only on *z* and the other terms do not so to be valid for all *z*, they must be related by a constant:

$$\frac{1}{R(\rho)}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)+\frac{1}{Q(\phi)}\frac{1}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial\phi^2}+k^2=0 \quad \text{where } k^2=\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$

- Multiply the left equation by ρ^2 :

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)+\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2}+k^2\rho^2=0 \quad \text{where } k^2=\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$

- The second term depends now only on *φ* and the rest of the terms do not, so we set it to a constant. All the partial derivatives become regular derivatives as the functions are of one variable only now:

$$\frac{\rho}{R(\rho)}\frac{d}{d\rho}\left(\rho\frac{d R(\rho)}{d\rho}\right)-\nu^2+k^2\rho^2=0 \quad \text{where} \quad -\nu^2=\frac{1}{Q(\phi)}\frac{d^2 Q(\phi)}{d\phi^2} \quad , \quad k^2=\frac{1}{Z(z)}\frac{d^2 Z(z)}{d z^2}$$

- We put each equation in a more intuitive form:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d R(\rho)}{d\rho}\right)+\left(k^2-\frac{\nu^2}{\rho^2}\right)R(\rho)=0 \quad \text{where} \quad \frac{d^2 Q(\phi)}{d\phi^2}=-\nu^2 Q(\phi) \quad , \quad \frac{d^2 Z(z)}{d z^2}=k^2 Z(z)$$

- In order to get the most general solution, we must cover all four cases:

- <u>Case 1</u>: If $\nu=0$ and $k=0$ then the solutions are:

$$R(\rho)=A_{0,0}+B_{0,0}\ln\rho \quad , \quad Q(\phi)=C_{0,0}+D_{0,0}\phi \quad \text{and} \quad Z(z)=F_{0,0}+G_{0,0}z$$

- The complete particular solution is the product of these parts.

- Case 2: If $\nu \neq 0$ and $k=0$ then the solutions are:

$$R(\rho)=A_{\nu,0}\rho^{\nu}+B_{\nu,0}\rho^{-\nu} \quad , \quad Q(\phi)=C_{\nu,0}e^{i\nu\phi}+D_{\nu,0}e^{-i\nu\phi} \quad \text{and} \quad Z(z)=F_{\nu,0}+G_{\nu,0}z$$

- The complete particular solution is the product of these parts.

- Case 3: If $\nu=0$ and $k\neq 0$, solutions to the last two equations are:

$$Q(\phi)=C_{0,k}+D_{0,k}\phi \quad , \quad Z(z)=F_{0,k}e^{kz}+G_{0,k}e^{-kz}$$

The differential equation involving *R* has the same solution as the next case and will be handled with it.

- Case 4: If $\nu\neq 0$ and $k\neq 0$, the solutions to the last two equations are:

$$Q(\phi)=C_{\nu,k}e^{i\nu\phi}+D_{\nu,k}e^{-i\nu\phi} \qquad Z(z)=F_{\nu,k}e^{kz}+G_{\nu,k}e^{-kz}$$

- The differential equation involving *R* must be solved by trying a series solution. Once found, the general solution for this case will have the form:

$$\Phi(\rho,\phi,z)=\sum_{\nu\neq 0}\sum_{k\neq 0} R_{\nu,k}(\rho)(C_{\nu,k}e^{i\nu\phi}+D_{\nu,k}e^{-i\nu\phi})(F_{\nu,k}e^{kz}+G_{\nu,k}e^{-kz})$$

- First simplify the equation by making the substitution: $x=k\rho$

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dR(x)}{dx}\right)+\left(1-\frac{\nu^2}{x^2}\right)R(x)=0$$

- Try a solution of the form $R(x)=\sum_{j=0}^{\infty} a_j x^{j+\alpha}$ and substitute it in:

$$\sum_{j=0}^{\infty} a_j x^{j+\alpha-2}\left((j+\alpha)^2-\nu^2\right)+\sum_{j=0}^{\infty} a_j x^{j+\alpha}=0$$

- Remove the first two terms of the sum on the left and combine the remaining sums:

$$a_0 x^{\alpha-2}\left(\alpha^2-\nu^2\right)+a_1 x^{\alpha-1}\left((1+\alpha)^2-\nu^2\right)+\sum_{j=0}^{\infty} x^{j+\alpha}\left(a_{j+2}\left((j+2+\alpha)^2-\nu^2\right)+a_j\right)=0$$

- Every power of *x* is independent, thus the coefficient of every power must vanish:

$$a_0\left(\alpha^2-\nu^2\right)=0 \quad , \quad a_1\left((1+\alpha)^2-\nu^2\right)=0 \quad , \quad a_{j+2}=-\frac{1}{\left((j+2+\alpha)^2-\nu^2\right)}a_j$$

- From these it is found that $\alpha=\pm\nu$, $a_{\text{odd}}=0$ and $a_{j+2}=-\frac{1}{(j+2)(j+2\pm 2\nu)}a_j$

- Because all odd powers are zero, we can iterate over all integers and rewrite the indices:

$$a_{2j} = -\frac{1}{4j(j \pm \nu)} a_{2(j-1)} \quad \text{for } j = 1, 2, 3, \ldots$$

- In summary, one solution to the *R* equation is the Bessel *J* function:

$R_{\nu,k}(\rho) = J_\nu(k\rho)$ where $J_\nu(x) = \sum_{j=1}^{\infty} a_{2(j-1)} x^{2(j-1)+\nu}$, $a_{2j} = -\frac{1}{4j(j+\nu)} a_{2(j-1)}$ and by convention, $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$ in terms of the Gamma function.

- One might think that the other solution to the *R* equation is $J_{-\nu}$, but it turns out that this is not linearly dependent from J_ν when *ν* is an integer and we must find another solution.
- Traditionally, the Neumann function, or Bessel function of the second kind is taken as the other linearly independent solution. It is defined by:

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu x)}$$

- Useful properties, including the recurrence relations, can be easily derived or found in textbooks.
- The general solution for *R*, in terms of the Bessel functions, becomes:

$$R_{\nu,k}(\rho) = A_{\nu,k} J_\nu(k\rho) + B_{\nu,k} N_\nu(k\rho)$$

- The general solution to the Laplace equation in cylindrical coordinates for case 4 becomes:

$$\Phi(\rho,\phi,z) = \sum_{\nu\neq 0}\sum_{k\neq 0}(A_{\nu,k}J_\nu(k\rho) + B_{\nu,k}N_\nu(k\rho))(C_{\nu,k}e^{i\nu\phi} + D_{\nu,k}e^{-i\nu\phi})(F_{\nu,k}e^{kz} + G_{\nu,k}e^{-kz})$$

- The most general solution to the Laplace equation in cylindrical coordinates is the sum of all the solutions of all the possible cases:

$$\boxed{\begin{aligned}\Phi(\rho,\phi,z) = {} & (A_{0,0} + B_{0,0}\ln\rho)(C_{0,0} + D_{0,0}\phi)(F_{0,0} + G_{0,0}z) \\ & + \sum_{\nu\neq 0}(A_{\nu,0}\rho^\nu + B_{\nu,0}\rho^{-\nu})(C_{\nu,0}e^{i\nu\phi} + D_{\nu,0}e^{-i\nu\phi})(F_{\nu,0} + G_{\nu,0}z) \\ & + \sum_{k\neq 0}(A_{0,k}J_0(k\rho) + B_{0,k}N_0(k\rho))(C_{0,k} + D_{0,k}\phi)(F_{0,k}e^{kz} + G_{0,k}e^{-kz}) \\ & + \sum_{\nu\neq 0}\sum_{k\neq 0}(A_{\nu,k}J_\nu(k\rho) + B_{\nu,k}N_\nu(k\rho))(C_{\nu,k}e^{i\nu\phi} + D_{\nu,k}e^{-i\nu\phi})(F_{\nu,k}e^{kz} + G_{\nu,k}e^{-kz})\end{aligned}}$$

- To be useful, we must find orthogonal sets of functions involving the Bessel functions so that we can use the orthogonality condition to invert equations involving the boundary conditions. The functions:

$$\sqrt{\rho}\, J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$$

can be shown to be orthogonal on the interval (0, *a*) for fixed *v* and $n = 1, 2, 3,...$ where x_{vn} are the roots that satisfy $J_\nu(x_{\nu n})=0$.

- The orthogonality condition for these functions can be shown to be:

$$\int_0^a \rho J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)d\rho=\frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2\delta_{n' n}$$

- This can be used to expand an arbitrary function in a Bessel series:

$$\boxed{f(\rho)=\sum_{j=1}^{\infty}A_{\nu n}J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)} \quad \text{where} \quad \boxed{A_{\nu n}=\frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})}\int_0^a \rho f(\rho) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)}$$

6. Boundary Value Problems in Cylindrical Coordinates

- Consider a cylinder with radius *a* and height *L* with its bottom centered at the origin. The potential is everywhere zero on the surface of the cylinder, except the top where it is a potential $V(\rho,\phi)$. This leads to the boundary conditions: $\Phi(\rho=a,\phi,z)=0$, $\Phi(\rho,\phi,z=0)=0$, and $\Phi(\rho,\phi,z=L)=V(\rho,\phi)$

- The general solution to the Laplace equation in cylindrical coordinates is:

$$\begin{aligned}\Phi(\rho,\phi,z)=&(A_{0,0}+B_{0,0}\ln\rho)(C_{0,0}+D_{0,0}\phi)(F_{0,0}+G_{0,0}z)\\ &+\sum_{\nu\neq 0}(A_{\nu,0}\rho^\nu+B_{\nu,0}\rho^{-\nu})(C_{\nu,0}e^{i\nu\phi}+D_{\nu,0}e^{-i\nu\phi})(F_{\nu,0}+G_{\nu,0}z)\\ &+\sum_{k\neq 0}(A_{0,k}J_0(k\rho)+B_{0,k}N_0(k\rho))(C_{0,k}+D_{0,k}\phi)(F_{0,k}e^{kz}+G_{0,k}e^{-kz})\\ &+\sum_{\nu\neq 0}\sum_{k\neq 0}(A_{\nu,k}J_\nu(k\rho)+B_{\nu,k}N_\nu(k\rho))(C_{\nu,k}e^{i\nu\phi}+D_{\nu,k}e^{-i\nu\phi})(F_{\nu,k}e^{kz}+G_{\nu,k}e^{-kz})\end{aligned}$$

- The region where we need a valid solution includes the origin, so several of the coefficients must be zero to keep the solution from blowing up. This leads to:

$$\begin{aligned}\Phi(\rho,\phi,z)=&(C_{0,0}+D_{0,0}\phi)(F_{0,0}+G_{0,0}z)\\ &+\sum_{\nu\neq 0}\rho^\nu(C_{\nu,0}e^{i\nu\phi}+D_{\nu,0}e^{-i\nu\phi})(F_{\nu,0}+G_{\nu,0}z)\\ &+\sum_{k\neq 0}J_0(k\rho)(C_{0,k}+D_{0,k}\phi)(F_{0,k}e^{kz}+G_{0,k}e^{-kz})\\ &+\sum_{\nu\neq 0}\sum_{k\neq 0}J_\nu(k\rho)(C_{\nu,k}e^{i\nu\phi}+D_{\nu,k}e^{-i\nu\phi})(F_{\nu,k}e^{kz}+G_{\nu,k}e^{-kz})\end{aligned}$$

- The region where we need a valid solution spans the full circle of angles for *φ*, so we require the function to be single-valued:

$$\Phi(\rho,\phi,z)=\Phi(\rho,\phi+2\pi,z)$$

- This equality must hold for all values of all of the independent variables, thus each term must match. This requirement leads to: $D_{0,0}=0$, $D_{0,k}=0$, and $\nu=m$ where $m=0,1,2,3,\ldots$

$$\begin{aligned}\Phi(\rho,\phi,z)=&(F_{0,0}+G_{0,0}z)\\&+\sum_{m\neq 0}\rho^m(C_{m,0}e^{im\phi}+D_{m,0}e^{-im\phi})(F_{m,0}+G_{m,0}z)\\&+\sum_{k\neq 0}J_0(k\rho)(F_{0,k}e^{kz}+G_{0,k}e^{-kz})\\&+\sum_{m\neq 0}\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})(F_{m,k}e^{kz}+G_{m,k}e^{-kz})\end{aligned}$$

- Apply the boundary condition: $\Phi(\rho,\phi,z=0)=0$

$$\begin{aligned}0=&F_{0,0}\\&+\sum_{m\neq 0}\rho^m(C_{m,0}e^{im\phi}+D_{m,0}e^{-im\phi})F_{m,0}\\&+\sum_{k\neq 0}J_0(k\rho)(F_{0,k}+G_{0,k})\\&+\sum_{m\neq 0}\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})(F_{m,k}+G_{m,k})\end{aligned}$$

- This leads to the requirements that $F_{0,0}=0$, $F_{m,0}=0$, $G_{0,k}=-F_{0,k}$, and $G_{m,k}=-F_{m,k}$

$$\begin{aligned}\Phi(\rho,\phi,z)=&G_{0,0}z\\&+\sum_{m\neq 0}\rho^m(C_{m,0}e^{im\phi}+D_{m,0}e^{-im\phi})z\\&+\sum_{k\neq 0}J_0(k\rho)F_{0,k}\sinh(kz)\\&+\sum_{m\neq 0}\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})\sinh(kz)\end{aligned}$$

- All the $m = 0$ terms now have a form for which they can be combined with the $m \neq 0$ terms:

$$\begin{aligned}\Phi(\rho,\phi,z)=&\sum_m\rho^m(C_{m,0}e^{im\phi}+D_{m,0}e^{-im\phi})z\\&+\sum_m\sum_{k\neq 0}J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})\sinh(kz)\end{aligned}$$

- The first term on the right is just a special case of the last term when $k = 0$ (close to zero, J_m has an asymptotic form that is proportional to ρ^m). They can be combined:

$$\Phi(\rho,\phi,z)=\sum_m\sum_k J_m(k\rho)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})\sinh(kz)$$

- Apply the boundary condition: $\Phi(\rho=a,\phi,z)=0$:

$$0=\sum_m\sum_k J_m(ka)(C_{m,k}e^{im\phi}+D_{m,k}e^{-im\phi})\sinh(kz)$$

- This can only hold true for all values of the independent variable ϕ and z if the coefficients are zero:

$0 = J_m(ka)$ for all m and k

- There is only a discrete set of solutions to this equations - its roots. The roots cannot be found analytically. When done numerically, the roots are designated as x_{mn} for n=1, 2, 3... so that

$$k = \frac{x_{mn}}{a}$$

- The solution now becomes:

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(x_{mn}\frac{\rho}{a}\right)(C_{m,n}e^{im\phi} + D_{m,n}e^{-im\phi})\sinh\left(x_{mn}\frac{z}{a}\right)$$

- The final boundary condition is $\Phi(\rho,\phi,z=L) = V(\rho,\phi)$ now applied:

$$V(\rho,\phi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(x_{mn}\frac{\rho}{a}\right)(C_{m,n}e^{im\phi} + D_{m,n}e^{-im\phi})\sinh\left(x_{mn}\frac{L}{a}\right)$$

- This is a Fourier series in ϕ and a Fourier-Bessel series in ρ. We find the coefficients in the usual way. Multiply both sides by an exponential and integrate:

$$\int_0^{2\pi} V(\rho,\phi)e^{-im'\phi}d\phi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(x_{mn}\frac{\rho}{a}\right)\int_0^{2\pi}(C_{m,n}e^{i(m-m')\phi} + D_{m,n}e^{i(-m-m')\phi})d\phi\sinh\left(x_{mn}\frac{L}{a}\right)$$

- Use the orthogonality condition $\int_{-\infty}^{\infty} e^{i(m-m')x}dx = 2\pi\,\delta_{m,m'}$ and relabel:

$$\int_0^{2\pi} V(\rho,\phi)e^{-im\phi}d\phi = 2\pi\sum_{n=1}^{\infty} J_m\left(x_{mn}\frac{\rho}{a}\right)C_{m,n}\sinh\left(x_{mn}\frac{L}{a}\right)$$

- Multiply both side by $\rho J_m\left(x_{mn'}\frac{\rho}{a}\right)$ and integrate:

$$\int_0^a \rho J_m\left(x_{mn'}\frac{\rho}{a}\right)\int_0^{2\pi} V(\rho,\phi)e^{-im\phi}d\phi\, d\rho = 2\pi\sum_{n=1}^{\infty}\int_0^a \rho J_m\left(x_{mn'}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho}{a}\right)d\rho\, C_{m,n}\sinh\left(x_{mn}\frac{L}{a}\right)$$

- Use the orthogonality condition: $\int_0^a \rho J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)d\rho = \frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2\delta_{n'n}$

$$\int_0^a \rho J_m\left(x_{mn'}\frac{\rho}{a}\right)\int_0^{2\pi} V(\rho,\phi)e^{-im\phi}d\phi\, d\rho = 2\pi\sum_{n=1}^{\infty}\frac{a^2}{2}[J_{m+1}(x_{mn})]^2\delta_{n'n}C_{m,n}\sinh\left(x_{mn}\frac{L}{a}\right)$$

$$\int_0^a \rho J_m\left(x_{mn}\frac{\rho}{a}\right)\int_0^{2\pi} V(\rho,\phi)e^{-im\phi}d\phi\, d\rho = \pi a^2[J_{m+1}(x_{mn})]^2 C_{m,n}\sinh\left(x_{mn}\frac{L}{a}\right)$$

- Solve for the coefficient:

$$C_{m,n} = \frac{\operatorname{csch}\left(x_{mn}\frac{L}{a}\right)}{\pi a^2 [J_{m+1}(x_{mn})]^2} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho\, V(\rho,\phi) J_m\left(x_{mn}\frac{\rho}{a}\right) e^{-im\phi}$$

- In the exact same process the other set of coefficients is found to be:

$$D_{m,n} = \frac{\operatorname{csch}\left(x_{mn}\frac{L}{a}\right)}{\pi a^2 [J_{m+1}(x_{mn})]^2} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho\, V(\rho,\phi) J_m\left(x_{mn}\frac{\rho}{a}\right) e^{im\phi}$$

- It is apparent now that $D_{m,n} = C^*_{m,n}$

- In summary, the final solution is:

$$\boxed{\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(x_{mn}\frac{\rho}{a}\right)\left(C_{m,n}e^{im\phi} + C^*_{m,n}e^{-im\phi}\right)\sinh\left(x_{mn}\frac{z}{a}\right)}$$

where $$\boxed{C_{m,n} = \frac{\operatorname{csch}\left(x_{mn}\frac{L}{a}\right)}{\pi a^2 [J_{m+1}(x_{mn})]^2} \int_0^{2\pi} d\phi \int_0^a d\rho\, \rho\, V(\rho,\phi) J_m\left(x_{mn}\frac{\rho}{a}\right) e^{-im\phi}}$$